Collegium of Economic Analysis

Summary of a doctoral dissertation

Field: social sciences

Discipline: economics and finance

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Fractal nature of price movements in financial markets

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TABLE OF CONTENTS

1. Introduction

This dissertation concerns the analysis of returns on financial instruments using methods related to the chaos theory. The starting point of the considerations undertaken in the dissertation was the efficient market hypothesis, based on the linear paradigm, the development of which has become an extremely important step in the field of economic science. This hypothesis was proposed by Eugene F. Fama (E. F. Fama, 1970, pp. 383-417) and is probably the most widely studied hypothesis within investment finance sciences. Its popularity is mainly due to the fact that it allows the use of probability calculus in the analysis of financial markets. The main assumption of the linear approach is that return rates in financial markets follow an approximately normal distribution and are independent of each other. The normal distribution is the basis of many statistical methods, but one of the main objectives of the paper was to show that the time series of financial instruments did not follow a normal distribution.

The first attempt to study return rates in financial markets in terms of their distribution was the work by Benoit B. Mandelbrot in 1963 (B. B. Mandelbrot, 1963, pp. 392-417). Its author showed that those distributions were left-asymmetric, meaning that a greater proportion of the density of the distribution lay in the left tail than in the right one. In addition, the tails were clearly thickened and the apex near the mean was higher than in normal distributions (this phenomenon is called leptokurtosis). For this reason, and due to the presence of many other deviations in the distributions of return rates on financial markets compared to a normal distribution (e.g.: volatility clustering, autocorrelation of return rates or the presence of long memory in volatility), it can be concluded that return rates on financial markets do not have a normal distribution (K. Piontek, 2005, pp. 297-308; E. E. Peters, 1997, p. 29).

One of the hypotheses concerning the nature of financial markets is the hypothesis that return rates on financial markets belong to a family of stable Pareto distributions (called fractal distributions), characterised by infinite variance, as already suggested by Mandelbrot in 1963 (B. B. Mandelbrot, 1963). A hypothesis that takes this approach into account is called the fractal market hypothesis and was proposed by Edgar E. Peters in 1991 (E. E. Peters, 1991). It is related to the research of Harold E. Hurst conducted in the first half of the 20th century, who deduced that most natural systems did not fit into the then prevailing concept of random walk, but that those systems were subject to a biased random walk. This theory is consistent with Peters' assumptions about the market, according to which globally (in the long term) the market is deterministic, while locally (in the short term), as a result of randomly occurring information and the emotional reactions of market participants to this information, it behaves in a random manner (E. E. Peters, 1997, pp. 45, 64).

At this point it is worth briefly explaining the concept of the fractal. According to Michał Tempczyk, there is no smallest particle of reality and no set of indivisible and homogeneous particles, since the primordial materials of reality divide infinitely "in depth" (M. Tempczyk, 1986, pp. 143-173). It follows that fractals (Latin *fractus* - fractured), generated by non-linear dynamical systems, are perfectly suitable for describing most natural shapes, and thus also time series, since "...fractal shapes are self-similar with respect to space, and fractal time series are characterised by statistical self-similarity with respect to time" (E. E. Peters, 1997, p. 49). The individual points of fractal time series are related to each other by mutual correlations that are not observed for random time series (M. J. Luczak, 2005, p. 98; P. Kowal, 2008, p. 6).

The concept of the chaos theory, which is now used in many fields of science, not only in finance or economics, is inextricably linked to the fractal market hypothesis. The chaos theory has many fathers, including, for example, Ilya Prigogine, Edward N. Lorenz or the aforementioned Mandelbrot, and its importance is compared to that of the theory of relativity and quantum mechanics from the beginning of the 20th century (M. Waszczyk, 2002, p. 2). One of the more characteristic features of the chaos theory is its universality. Indeed, its concepts are applied and used in many branches of technical, natural or social sciences. Given the subject matter addressed in the dissertation, the most interesting application of the chaos theory is in economics and finance. According to Jakimowicz, "...numerical studies of archetypal economic models allow us to conclude that economic systems have a natural tendency to move towards the edge of chaos, i.e. a transitional state between order and chaos, where complexity is optimal" (A. Jakimowicz, 2010, p. 253). A system in this sense is a collection of elements between which certain relationships and interactions take place. It should be mentioned that the necessary conditions for the occurrence of chaos include: non-linearity – the system must be described by non-linear differential equations or differential equations, and openness of the system – which means that at least one of the system parameters must depend on external factors (A. Jakimowicz, 2005, pp. 321-322).

In non-linear dynamic systems, randomness and determinism coexist to create a static order. Although in these systems the outcomes appear to be random, they are the result of specific laws (N. Siemieniuk, T. Siemieniuk, 2015, p. 182). This combination of local randomness and global order contributes to dynamic processes that are resilient to external influences and are well adapted to changing conditions (E. E. Peters, 1997, pp. 43-44; T. Velásquez, 2010, pp. 235-236). This explains the constant evolution of markets and is probably the reason why no one has yet succeeded in developing a reliable investment method that provides consistently high above-average return rates over a long-term horizon (M. N. Rothbard, 2015, p. 24).

Summarising the issues described above, typical attributes of non-linear dynamic systems can be observed. According to Peters, if financial markets are such systems, they are characterised by the following properties (E. E. Peters, 1997, pp. 8-10):

- the existence of long-term correlations and trends, which is a feedback effect,
- "whimsical" behaviour of markets under certain specific conditions and periods, which involves the existence of critical levels at which more than one equilibrium state exists,
- the existence of time series of return rates, characterised by self-similarity and fractal structure,
- decreasing accuracy of forecasts as their time horizon increases, due to the very high sensitivity of the system to a change in initial conditions.

2. Justification of the choice of the subject

For several decades, authors of scientific studies have been investigating various problems related to the broader economy, improving numerous methods through their various modifications. The fundamental problem seems to be the complexity of economic phenomena and the large number of variables affecting non-linear dynamic systems. Available research results often show inconclusive results, sometimes varying depending on the methods used.

For the author of this dissertation it was most interesting to look at the application of the chaos theory methods to the analysis of financial time series. The motivation for investigating the issue in question was precisely the complexity of economic phenomena, which are so difficult to analyse and interpret that – although many investors seek to forecast financial time series – it is still extremely difficult, if at all possible in the long term. Since the market is very dynamic and far from equilibrium, it made sense to investigate the extent to which order and regularities in the time series of financial instruments could be detected, which could be useful in terms of forecasting and therefore also in investment terms.

The emergence of successive crises has forced researchers to look at the functioning of the market in a different way. Indeed, the temporal feedback mechanism is another important feature of non-linear dynamic systems (E. E. Peters, 1997, pp. 4-6; A. Jakimowicz, 2013, p. 363). The extension of the issues of the chaos theory to the economic sciences has significantly influenced the change in thinking about the methods of analysis of economic activity and the explanation of many phenomena concerning fluctuations, instability, crises and recessions (M. Faggini, A. Parziale, 2012, p. 4). The issue of the impact of abnormal phenomena on financial time series has become an important element of the work, as it is precisely such events that make it difficult, and often impossible, to forecast financial time series in the long term.

According to Peters (1997), financial time series are characterised by deterministic chaos and natural cycles of the system, and in the analysis of such series – in his view considered non-linear - time is extremely important in the forecasting process. Unlike random time series, where the number of data in the sample is the most important factor in the analysis, in this case it is of little importance, but the time series must be long enough to have a minimum of ten natural cycles. During the literature review, were noticed few sources drawing attention to this problem. Most authors do not address the topic of natural cycles at all, or if they do, they only delineate the cycle length of a given time series, but do not take into account the need to determine the necessary number of cycles so that the results of the study are subject to as little error as possible. This issue also seemed interesting enough to be investigated and analysed as part of a doctoral dissertation.

3. Aims of the paper, research questions and hypotheses

The dissertation formulated three thesis objectives, two research questions and two main and four specific research hypotheses. A summary of these is presented in the following table (Table 1).

Table 1. *Aims of the paper, research questions and hypotheses.*

Aims of the paper	Wording
Aim 1	Verification of the assumption of efficiency of financial markets.
Aim 2	An attempt to demonstrate the fractal nature of price movements in financial markets.
Aim 3	Empirical analysis of diverse financial instruments using a variety of research methods.
Research questions	**Wording**
Question 1	Are financial markets efficient or fractal?
Question 2	Do price movements in the financial markets have the nature of a random walk or a biased random walk?
Research hypotheses	**Wording**
Main hypothesis 1	Financial markets are not efficient but fractal.
Main hypothesis 2	Price movements in financial markets are not of a random walk nature, but are of a biased random walk nature: the graphs of financial time series trajectories are fractals.
Specific hypothesis 1	Distributions of return rates of time series of financial instruments are not normal.
Specific hypothesis 2	The time series of financial instruments are characterised by self-similarity (statistical self-similarity).
Specific hypothesis 3	Financial time series are characterised by a long-term memory effect.
Specific hypothesis 4	Financial time series are difficult to forecast due to the very high sensitivity of non-linear dynamic systems to a change in initial conditions.

Source: *author's own study.*

4. Structure of the dissertation

The dissertation consists of 371 pages and has been planned to be as coherent and readable as possible. The dissertation is divided into four main parts, of which the first three contain theoretical issues and the fourth part, the most comprehensive one, is a presentation of the empirical research with conclusions. The structure of the work is presented in the table below, where the names of the chapters within each part are also listed (Table 2).

Table 2: *Structure of the dissertation*

Structure of the dissertation	Content
Introduction	
Part I	**Financial markets in the context of linearity**
Chapter 1	Financial markets in terms of the linear paradigm
Chapter 2	Financial markets in terms of the chaos theory
Chapter 3	Long-term memory occurring in non-linear dynamic systems
Chapter 4	Non-linear dynamic systems in finance
Part II	**Fractals and elements of fractal geometry**
Chapter 1	Introduction to the fractal theory
Chapter 2	Self-similarity
Chapter 3	Deterministic fractals
Chapter 4	Random fractals
Chapter 5	Multifractals
Chapter 6	Types of transformations
Chapter 7	Dimension of geometrical objects
Part III	**Empirical research methods**
Chapter 1	Introduction
Chapter 2	Analysis of financial variables using irregularity measures
Chapter 3	Analysis of financial variables using Brownian motions
Chapter 4	Verification of empirical data by searching for the natural cycle of a dynamic system
Chapter 5	Investigating the incidence of long-term memory of time series using a data shuffle test
Chapter 6	Analysis of financial markets using measures of system sensitivity to changing initial conditions
Chapter 7	Selected literature sources providing evidence of deterministic chaos in financial time series
Part IV	**Presentation and analysis of empirical research results**
Chapter 1	Introduction
Chapter 2	Interpretation of the obtained research results in terms of financial analysis
Chapter 3	Analysis of selected stock indices
Chapter 4	Analysis of selected shares
Chapter 5	Analysis of selected bonds
Chapter 6	Analysis of selected raw materials
Chapter 7	Analysis of selected currency pairs
Chapter 8	Summary of the obtained research results
Chapter 9	Author's own contribution
Chapter 10	Literature review in terms of the verification of the obtained

Structure of the dissertation	Content
	results
Summary and conclusions	
Attachments	
Attachment 1	Examples of deterministic fractals created iteratively, called geometric fractals
Attachment 2	Examples of deterministic fractals defined by a recursive relation of space points, called algebraic fractals
Attachment 3	Euler's Gamma function
Attachment 4	Images of selected fractals (colour inset)
List of figures	Part I – 17, Part II – 17, Part III – 11, Part IV – 77
Table of tables	Part I – 0, Part II – 1, Part III – 3, Part IV – 30
List of English phrases	
Literature sources	Compact literature – 192, Websites – 20
Summary	
Abstract	

Source: *author's own study.*

5. Research sources and methods

The paper was written based on Polish and English-language literature sources and using websites related to the given field. A total of 212 sources were used (see Table 2). The literature on the application of research methods based on the chaos theory in the field of economics and finance is quite rich and has been successively expanded for several decades.

Recently, many new methods for time series analysis have been introduced into economics. Analyses to detect non-linearity and deterministic chaos in economic time series are most often complex in nature. Due to the difficulty of detecting all the variables affecting the system under study, different methods are used and then the results obtained are compared. Thus, we can speak of the birth of a new field – qualitative econometrics, which studies the complexity of economic dynamic systems, while it should be mentioned that it is precisely economic systems that are considered to be among the most complex (A. Jakimowicz, 2010, pp. 240-241). The greatest difficulties in identifying chaos in non-linear dynamic systems are considered to be the sensitivity to:

the number of data, the values of parameters considered in the analysis, excessive aggregation of data and the presence of random noise in the data. It is also more difficult to analyse systems characterised by high dimensionality. In fact, the detection of chaos usually first looks for two important chaotic features in the system: determinism and irregularity of the dynamics of a given system (W. Orzeszko, 2016, pp. 379-380).

In order to obtain the most reliable empirical results possible, it was decided to use a wide variety of research methods (9 in total) to analyse the selected financial time series, such as:

1) The Hurst exponent.
2) The pointwise Hölder exponents.
3) Fractional Brownian motions.
4) Multifractional Brownian motions.
5) Exploring the natural cycle of a dynamic system.
6) Investigating the incidence of long-term memory of time series using a data shuffle test.
7) The fractional dimension.
8) Lyapunov exponents.
9) Bifurcation diagram.

Hurst exponent (Q, Bui, R. Ślepaczuk, 2022, p. 2; H. E. Hurst, 1951, pp. 770-799; R. Kutner, 2009, p. 25):

The most important turning point in the field of performing long-range analysis of time series was undoubtedly the development by Harold Edwin Hurst in 1951 of the method of re-scaled range analysis R/S (this is the average distance that the value of a time series moves away from its starting point, divided by the standard deviation) and the Hurst exponent he proposed (which makes it possible to distinguish between random series and series with a biased random walk, as well as to indicate the sign and degree of correlation of the data). The Hurst exponent estimates are the basis for a classification according to their values and inferring the characteristics that the analysed series may have:

$0 < H < 0.5$ Negative correlation (antipersistent series).

$H = 0.5$ No correlation (random series).

$0.5 < H < 1$ Long-range correlation (persistent series).

The pointwise Hölder exponents (A. Mastalerz-Kodzis, 2003, pp. 49-51; S. Seuret, J. Lévy-Véhel, 2002, p. 263):

Using the pointwise Hölder exponents, one can study the complexity of functions and the trajectories of certain stochastic processes in the neighbourhood of an arbitrary point. By definition, the Hölder function is continuous over the entire domain and its graph is fractal in nature. A generalized Hölder function is discontinuous and may be continuous in intervals. The Hölder function in different intervals has the characteristics of different random walk processes. It can be an antipersistent series in some places and a persistent series in others.

The interpretation of the Hölder exponent at a point is the same as that of the Hurst exponent, except that the Hurst exponents apply to the global analysis of the series and the Hölder exponents apply to the local analysis.

Fractional Brownian motions (B. Mandelbrot, 1972, p. 266; E. E. Peters, 1997, pp. 63-64):

The concept of fractional Brownian motions was born as a response to criticism of the market efficiency hypothesis. A fractional Brownian motion process, characterised by self-similar increments, can have an infinite positive correlation when H>0.5 or an infinite negative correlation when H<0.5, while it takes the form of a standard Brownian motion process when H=0.5.

H close to 1→ the graph is a random curve almost smooth.

H close to 0→ the graph becomes an irregular random curve.

For H=0→ the standard Brownian motion process.

For H=1 → the process is a deterministic process.

Multifractional Brownian motions (R. Kutner, 2009, p. 9; A. Mastalerz-Kodzis, 2003, p. 80):

The greatest advantage of multifractional Brownian motions is the possibility to study the irregularity of time series in different time intervals. To do this, one should replace the Hurst exponent *H* (constant for the whole process) by a time-dependent function *H(t)*, thus obtaining a local magnitude in time from a global magnitude. The function *H(t)* is called the Hölder function or the local degree of self-similarity, and the process is referred to as a multifractal process or a multifractional Brownian motion process.

When the values of the Hölder function are close to 0, the graph is characterised by a zigzag structure. The closer the function values are to 1, the smoother the multifractional Brownian motion process is. The interpretation of the multifractional Brownian motion is similar to that of the fractional Brownian motion, with the features of the fractional Brownian motion process in this case being considered only locally.

The dynamic system cycle (R. Buła, 2015, pp. 85-86; W. Orzeszko, 2016, pp. 41-42; E. E. Peters, 1997, pp. 84-104):

During the analysis of key economic and stock market variables, the occurrence of approximately 4-5 year non-periodic cycles was observed. The length of a cycle is a measure of the time it takes for the impact of single period events to become undetectable. Even extreme events such as wars, crises or stock market crashes do not change this. A re-scaled (R/S) range analysis can be used to find the cycle length of dynamic systems. The cycle length is determined by the point of memory loss, from which the graph starts to run along a random walk line. To estimate the average length of one cycle, linear regression should be applied to a series of re-scaled ranges for successive values representing the number of observations. It follows from the chaos theory that a sufficient number of data is considered to be one which covers ten cycles. The cycles that occur are independent of the density of the data, so if the number of observations is too small for the phenomenon under study, even the adoption of daily, hourly or minutely data in the analysis will not improve the situation, because in this case time is more important than the number of data.

The fractional dimension (J. Kudrewicz, 2007, p. 20; A. Mastalerz-Kodzis, pp. 42-43; E. E. Peters, 1997, pp. 58-59):

Dimension expresses the way in which a geometric object (or time series) fills space. The dimension of a time series is a measure of its zigzagging. In the case of persistent time series, the fractional dimension is closer to the dimension of a straight line and the closer to unity the value of the Hurst exponent is, the smoother and less jagged the line of the graph is compared to a random walk. A fractal time series is not purely deterministic. Such a system is also influenced by the more so, the larger than 1 the fractional dimension is. Depending on the value of the Hurst exponent, the following cases can be specified:

if $0.5<H<1$, then $1<D<1.5$ (persistent time series, i.e. trend enhancing), if H=0.5, then D=1.5 (random walk),

if $0<H<0.5$, then $1.5<D<2$ (antipersistent time series).

The positive Lyapunov exponent (M. Faggini, A. Parziale, 2012, p. 6; A. Jakimowicz, 2013, p. 365; W. Orzeszko, 2016, p. 156):

Lyapunov exponents are the most commonly used quantitative measure of divergence, or the divergence of adjacent trajectories (orbits) in phase space. They determine the sensitivity of the system to changes in initial conditions, which causes the predictability to decrease exponentially as the forecast horizon lengthens. A necessary condition for an m-dimensional system to be chaotic is to have at least one positive Lyapunov exponent. However, the existence of positive Lyapunov exponents is not a sufficient condition to confirm the existence of deterministic chaos in time series.

In economic practice, determining the future behaviour of a system is an important issue. The existence of a positive Lyapunov exponent makes it impossible to effectively forecast the evolution of a single state over the long term.

Bifurcation diagram (J. Avrejcewicz, 1997, p. 31; E. E. Peters, 1997, pp. 125-128):

The bifurcation diagram is a simple non-linear model showing the strength of chaotic behaviours. The transition from a state of order to chaos involves rapid qualitative changes, known as bifurcations. The occurrence of deterministic chaos is indicated by the 'stability band' (the white wide band in the diagram), where smaller copies of the main shape are observed. The higher the value of the Hurst exponent, the more areas in the diagram show self-similarity (Figure 1). A bifurcation diagram is a set of possible solutions to an equation (there are an infinite number of solutions in a finite space).

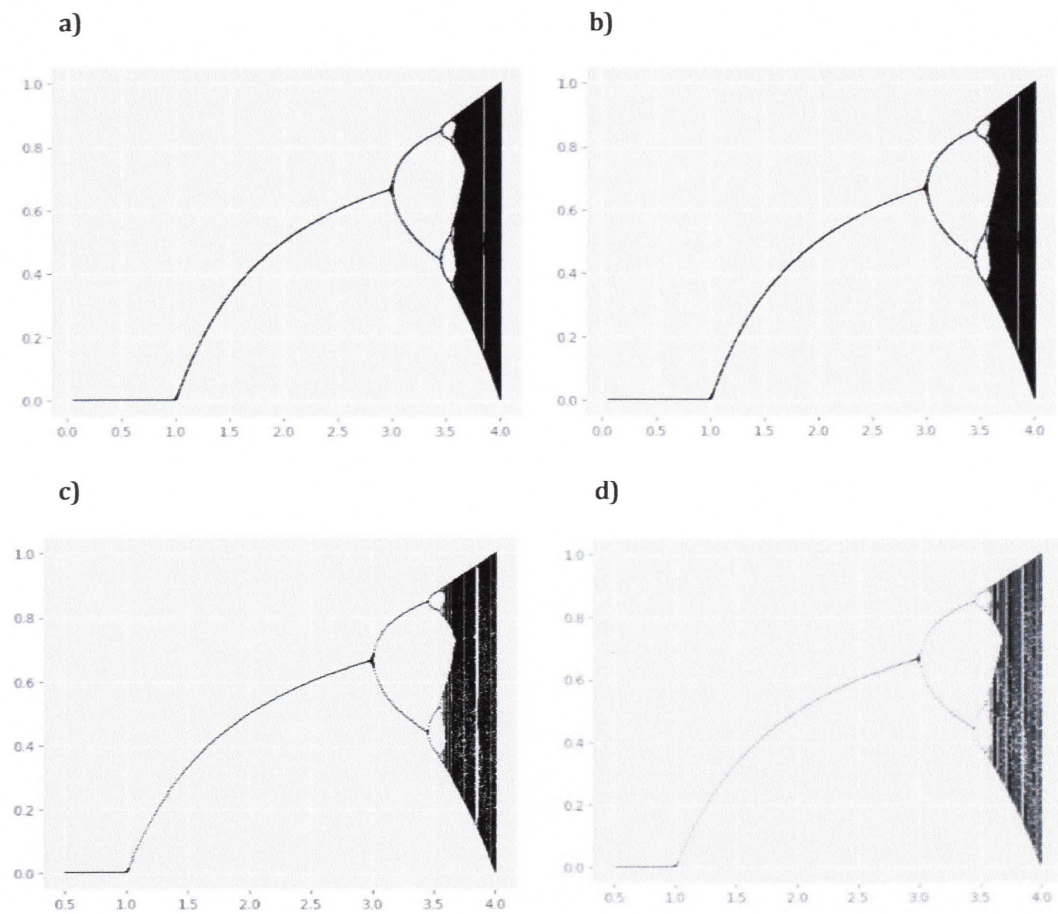

***Figure 1**: Bifurcation diagrams with highlighted stability bands indicating deterministic chaos: a) random series with H=05, b) IBM share series with H=0.56, c) S&P500 series with H=0.63, d) DOW JONES series with H=0.68. D. Source: author's own study.*

6. Description and results of the conducted research

Conducting empirical research required deciding on the time range. Due to the occurrence of the global COVID-19 pandemic in the recent past and a subsequent crisis event shortly thereafter, i.e. the full-scale war in Ukraine, three time ranges were included in the analysis:

- from the adopted start of measurement until 31.12.2019,
- from the adopted start of measurement until 31.12.2021,
- from the adopted start of measurement until 31.12.2023.

The following financial instruments (17 in total) were analysed:

The following financial instruments (17 in total) were analysed:

Stock indices:

- Europe: WIG20 (Poland), DAX (Germany), SMI20 (Switzerland).
- America: S&P500 (USA), DOW JONES (USA), IPC (Mexico).
- Asia: HANG SENG (Hong Kong), NIKKEI225 (Japan), BSE30 (India).

Shares: COCA COLA (USA), MICROSOFT (USA).

Bonds: US 10Y BOND YIELD (USA), US 30Y BOND YIELD (USA).

Raw materials: XAU/USD, COPPER/USD.

Forex market: currency pairs: USD/PLN, GBP/USD.

All calculations were performed using the Python language (Anaconda 3.11.7) with the functions of the pandas (https://pandas.pydata.org/) and numpy (https://numpy.org/) libraries. The results of the calculations were formatted using Excel. Data for the analysis were obtained from the yfinance library (https://pypi.org/project/yfinance/) and some data were downloaded from https://stooq.pl/.

A summary table showing the extent to which the presence of deterministic chaos in selected sample financial time series was confirmed using the methods in question is presented below (Table 3). It was assumed that if at least 6 out of 9 methods confirmed the fractal nature of a time series, it could be considered to have the characteristics of a biased random walk (i.e. the graphs of the trajectories of the financial time series data are fractals).

***Table 3.** Occurrence of deterministic chaos in financial series according to individual methods (YES - confirmation, NO - negation, N/W - not known).*

	Hurst exponent	Hölder pointwise exponents	Fractional Brownian motions	Multifractional Brownian motions	Dynamic system cycle	Data shuffle test	Fractional dimension	Positive Lyapunov exponent	Bifurcation diagram	**Time series with deterministic chaos**
WIG20 (Poland)	N/W	YES	YES	YES	N/W	N/W	YES	YES	YES	**YES**
DAX (Germany)	YES	YES	YES	YES	YES	YES	YES	YES	NO	**YES**
SMI20 (Switzerland)	YES	YES	YES	YES	YES	YES	YES	YES	NO	**YES**
S&P500 (USA)	YES	YES	YES	YES	YES	YES	YES	YES	YES	**YES**
DOW JONES (USA)	YES	YES	YES	YES	YES	YES	YES	YES	YES	**YES**
IPC (Mexico)	N/W	YES	NO	NO	YES	YES	YES	YES	NO	**NO**
HANG SENG (Hong Kong)	NO	YES	NO	NO	NO	NO	N/W	YES	NO	**NO**
NIKKEI225 (Japan)	YES	YES	YES	YES	YES	YES	YES	YES	NO	**YES**
BSE30 (India)	YES	YES	YES	YES	YES	YES	YES	YES	YES	**YES**
Coca Cola (USA)	YES	YES	YES	YES	YES	YES	YES	YES	YES	**YES**
Microsoft (USA)	YES	YES	YES	YES	YES	YES	YES	YES	YES	**YES**
US 10Y BOND YIELD (USA)	NO	YES	YES	YES	NO	NO	N/W	NO	NO	**NO**
US 30Y BOND YIELD (USA)	NO	YES	YES	YES	NO	NO	N/W	YES	NO	**NO**
XAU/USD	YES	YES	YES	YES	YES	YES	YES	YES	NO	**YES**
COPPER/USD	YES	YES	YES	YES	YES	YES	YES	YES	NO	**YES**
USD/PLN	YES	YES	N/W	N/W	NO	NO	YES	YES	NO	**NO**
GPB/USD	YES	YES	NO	NO	NO	NO	YES	NO	NO	**NO**

***Source:** author's own study.*

After the conducted analysis, it can be concluded that most of the financial time series studied (as many as 11/17) have features of a biased random walk. The remaining time series have some features indicating their fractal nature, but the number of such features is too small, so they were considered to be random time series. However, it should be borne in mind that the dissertation shows that some features of deterministic chaos were observed locally in all the time series studied. It is also worth noting the ambiguity of the results obtained for some financial instruments. Difficulties in interpretation may be the result of the regulatory policies of governments and central banks, which significantly affect the prices of the indices in question and, therefore, the results of the conducted research.

Analysis of the data depending on the adopted range (up to the end of 2019, 2021 and 2023) highlighted the different results obtained depending on the adopted time horizon. By carrying out time series analyses in three different ranges, it was noted that for most of the indices considered, the values of the Hurst exponents were lower and lower with each successive range, so it can be concluded that abnormal events (e.g. wars, crises, crashes or even, in this case, the COVID-19 pandemic) have the effect of lowering the Hurst exponent and therefore increasing investment risk.

7. Summary and directions for further research

In conclusion, it can be considered that the obtained results confirm in the most cases the existence of long-term memory in financial time series, so the market is not efficient and the values of the financial instruments in question do not reflect all the information available on the market. On the basis of such time series, it can be concluded that price movements in financial markets do not have the character of a random walk but prices move according to a trend (price movements have the character of a biased random walk), and therefore the graphs of the trajectories of these time series are fractals, which confirms the main hypotheses of the dissertation.

The approaches and methods of research and analysis of financial time series presented in this paper certainly do not exhaust all possibilities. This is because the chaos theory offers an immeasurable number of different research methods, and it is likely that new, more refined measures will emerge over time to identify deterministic chaos in financial time series. The considerations undertaken in this dissertation are, in a way, complementary to those undertaken to date and provide a basis for further research in the field. Due to the complexity of the issues involved and the difficulties of interpretation, the obtained results should be approached with great caution.

It would be interesting to repeat a similar study in a decade or so, when most stock market indices, including those of developing countries, will have time series long enough to have a minimum of ten natural cycles of the dynamic system. It would also be worthwhile at that time to investigate how the selected time series would behave in a comparative analysis if they were assumed to be of comparable length, as the dissertation assumed time series of varying lengths, depending on the available data. Another direction worth considering is to prepare studies showing the impact of crisis events on financial time series, but taking into account similar time ranges and similar latitude or economic links, as local conditions affect the degree to which certain crises affect the financial instruments in question.

8. Literature cited in the summary

[1] Awrejcewicz J., *Secrets of nonlinear dynamics*, Wydawnictwo Politechniki Łódzkiej, Łódź 1997.

[2] Bui Q., Slepaczuk R., Applying Hurst Exponent in pair trading strategies on Nasdaq 100 index, *Physica A, no. 592*, 2022.

[3] Buła R., *Fluctuations of agricultural commodity prices in the light of fractal analysis, Challenges of modern economy - theoretical and practical aspects*, Publishing House of the Warsaw University of Life Sciences, monograph, Warsaw 2015.

[4] Faggini M., Parziale A., The failure of economic theory. Lessons from chaos theory, Scientific Research, *Modern Economy, no. 3*, 2012.

[5] Fama E. F., Efficient capital markets: a review of theory and empirical work, *Journal of Finance, vol. 25, no. 2*, 1970.

[6] Grobys K., A Fractal and comparative view of the memory of Bitcoin and S&P 500 returns, ELSEVIER, *Research in International Business and Finance, no. 66*, 2023.

[7] Hurst H. E., Long term storage capacity of reservoirs, *Transactions of the American Society of Civil Engineers, no. 116*, 1951.

[8] Jakimowicz A., Disasters and chaos in explaining the complexity of economic processes, *Studia Ekonomiczne, Institute of Economic Sciences, Polish Academy of Sciences, no. 3 (LLXXVIII)*, 2013.

[9] Jakimowicz A., *Od Keynes do teorii chaosu: ewolucja teorii wahań koniunkturalnych*, Polskie Wydawnictwo Naukowe, Warszawa 2005.

[10] Jakimowicz A., *Sources of instability of market structures*, Polskie Wydawnictwo Naukowe, Warsaw 2010.

[11] Kowal P., *What is the length of the Baltic coast? Measurement of fractal structures*, Wrocław University of Technology, unpublished engineering thesis, Wrocław 2008.

[12] Kudrewicz J., *Fractals and chaos*, Wydawnictwa Naukowo-Techniczne, Warsaw 2007.

[13] Kutner R., *Computer simulations of syngular and singular processes in finance – selected algorithms* [pdf], Warsaw 2009 [Retrieved from: https://www.fuw.edu.pl/ tl_files/studies/materials/ef/Hurst_Finance.pdf].

[14] Łuczak M. J., Application of fractals to obtain knowledge about capital markets, *Polish Association of Knowledge Management, no. 4*, Bydgoszcz 2005.

[15] Mandelbrot B. B., Statistical methodology for non-periodic cycles: from the covariance to R/S analysis, *Annals of Economic and Social Measurement, no. 1 (3)*, 1972.

[16] Mandelbrot B. B., The variation of certain speculative prices, *Journal of Business, no. XXXVI*, 1963.

[17] Mastalerz-Kodzis A., *Modelling processes on the capital market using multifractals*, Publishing House of the Academy of Economics in Katowice, Katowice 2003.

[18] Orzeszko W., *Nonparametric identification of nonlinearities in financial and economic time series*, Scientific Publishing House of the Nicolaus Copernicus University, Toruń 2016.

[19] Peters E. E., *Chaos and order in the capital markets: a new view of cycles, prices, and market volatility*, Wiley, New York 1991.

[20] Peters E. E., *Chaos theory and capital markets*, WIG-PRESS, Warsaw 1997.

[21] Piontek K., Modelling properties of rate of return series – skewness of distributions, *Prace Naukowe Akademii Ekonomicznej we Wrocławiu. Econometrics, no. 15 (1096)*, 2005.

[22] Rothbard M. N., *An economic point of view*, Ludwig Von Mises Institute, Wrocław 2015.

[23] Sánchez M. A., Trinidad J. E., García J., Fernández M., The effect of the underlying distribution in Hurst exponent estimation, *PLOS ONE*, 2015.

[24] Siemieniuk N., Siemieniuk T., Deterministic chaos theory and decisions of stock market investors, *Zeszyty Naukowe Uniwersytetu Szczecińskiego. Finance, Financial Markets, Insurance, vol. 1, no. 74*, Szczecin 2015.

[25] Tempczyk M., *Physics and the real world: elements of the philosophy of physics*, Polskie Wydawnictwo Naukowe, Warsaw 1986.

[26] Waszczyk M., The influence of chaos theory on some traditional ontological positions and on the dispute over reductionism, *Zeszyty Naukowe Politechniki Gdańskiej, vol. 6, no. 589*, Gdańsk 2002.

[27] Velásquez T., Chaos theory and the science of fractals in finance, *Copenhagen Business School, ODEON, no. 5*, 2010.