WARSAW SCHOOL OF ECONOMICS

Collegium of Economic Analysis

Optimal structure of a life insurance portfolio to hedge against longevity risk

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Abstract of the Ph.D. thesis prepared under the supervision of dr hab. Mariusz Skałba

Warsaw, March 2018

Introduction

The aim of the dissertation is to analyse life insurer's possibilities to benefit from natural immunization of their portfolio against changes in the level of mortality of the policyholders. Mortality and longevity risks belong to the main risks that a life insurance company is exposed to when conducting their activities. Mortality risk, being the risk that the actual mortality in the insured population differs - due to random effects - from the expectations, can be, to a large extent, limited by assuring sufficiently large and diverse insurance portfolio. Longevity risk, however, in its nature is related to persistent changes in the expected level of mortality, which usually have similar impact on the entire portfolio, thus no diversification can mitigate it. The longevity phenomenon can be illustrated with the changes of life expectancy in Poland. According to life tables, published by the Statistical Information Centre in Poland (GUS) within 27 years since 1990 up to 2016, the life expectancy at the moment of birth increased by 12% for males (from 66 to 74 years) and by 9% for females (from 75 to 82 years)[1]. The changes are even more evident, when looking at the life expectancy for older people, e.g. life expectancy for a 60 year old man increased by 26% (from 15 to 19 years) and for 60 year old woman by 23% (from 20 to 24 years) [2]. Consequently, in the case of long term products sold by life insurance companies the risk related to the changes in the level of mortality can be significant.

The problem of life insurers hedging against longevity risk affects mainly annuity products, i.e. products, where an insurance company is obliged to settle regular payments in a specified amount to the policyholder until their death. The level of annuity payments is calculated based on given demographic assumptions and a significant deviation between the actual and expected mortality can lead to severe losses of the insurance company. Currently in Poland there is no significant market for annuity products sold by insurance companies, thus the problem of hedging against longevity risk is not really present in daily activities of Polish insurers. Significantly higher exposure to this risk can be observed in insurance companies operating in many countries in Western Europe or USA, as they often offer private pension system, complementary to the public one. Nonetheless, it is a problem, which, especially in recent years, poses a great challenge for pension systems.

The scientific research presents different solutions to the problem of hedging life insurance portfolio against longevity risk. One of them is to purchase special hedging

[1] Data based on mortality tables published by the Statistical Information Centre in Poland (GUS) on the website: http://stat.gov.pl/obszary-tematyczne/ludnosc/trwanie-zycia/trwanie-zycia-tablice,1,1.html.
[2] *Ibidem*

instruments on the financial markets, such as longevity bonds or longevity swaps[3], for which payments are dependent on a specified mortality index. It is a good way of hedging for an insurance company, nevertheless it requires significant costs to purchase adequate instruments, and additionally no guarantee can be given that the instruments will fit the specific portfolio of the insurer. A second solution is to build a model forecasting future mortality coefficients and use it for the valuation of insurance products. The most popular models are Lee-Carter (LC)[4] model and stochastic Cairns-Blake-Dowd (CBD)[5] model. Such models can help insurance companies to estimate future payments more precisely and thanks to this decrease the risk. Nonetheless, both the calibration of the model parameters and forecasting based on the estimated models is subject to errors. Due to that, the actual mortality coefficients in the future can be significantly different to the forecast, thus the longevity risk that the insurance company is exposed to cannot be fully mitigated. A third and final solution to hedge against longevity risk is to benefit from natural immunization resulting from the adequate structure of the insurance portfolio. Different insurance products react to mortality changes in the population of policyholders in a different way. For protection products, such as life insurance, a decrease of mortality coefficients leads to lower payments and consequently to additional gain for the insurance company. For annuity products, on the contrary, increasing life expectancy implies that payments are settled for a longer period, which is a loss for the insurance company. The idea is to choose a proportion between different products in the portfolio in such a way, that losses on some products are offset by gains on other products making the entire portfolio insensitive to the changes of mortality level. However, it is not easy as the value of the insurance portfolio is determined by mortality coefficients, which are different for different age groups and there are many different scenarios how they can develop in the future. When trying to determine the optimal portfolio structure it is not possible to account for all possible scenarios of mortality changes and it is necessary to adopt some simplifying assumptions.

Literature review

The problem of natural immunization of life insurance portfolio by finding an optimal structure of the products is quite new for scientific research. First articles dealing with this

[3] Diverse financial instruments to hedge against longevity risk are presented in: D. Blake, M. Morales, *Longevity Risk and Capital Markets: The 2014-15 Update*, Journal of Risk and Insurance 84, p. 279-297 (2017).
[4] R. Lee, L. Carter, *Modeling and forecasting U.S. mortality*, Journal of the American Statistical Association 87, p. 659-671 (1992).
[5] A.J.G. Cairns, D. Blake, K. Dowd, *A Two-Factor Model for Stochastic Mortality with Parameter Uncertainty: Theory and Calibration*, The Journal of Risk and Insurance 73, p. 687-718 (2006).

problem were published in the first decade of the current century. It is worth to mention an article published by Samuel H. Cox and Yijia Lin in 2007. They proved the hypothesis that natural immunization is reflected in the real prices of products offered by insurance companies[6]. Based on the data on the level of life annuity single premium quoted by different insurance companies for a 65 year old man in years 1995-1998 the authors estimated an econometric model, which showed that higher share of life insurance policies in the insurer's portfolio led to lower price quoted for annuities. This discovery was an evidence that natural immunization indeed exists in the insurance companies' activities. It did not, however, answer the question how to determine the optimal portfolio structure, which would minimize the exposure to longevity risk. The answer to this question was later given by J. L. Wang, H. C. Huang, S. S. Yang and J. T. Tsai in 2010, when they published the results of their analyses, presenting the idea to determine the optimal portfolio structure based on so called mortality duration[7]. The idea of this approach was to set the proportions of life insurance and annuity policies (assuming no other products sold) in such a way that the duration corresponding to the sensitivity of the portfolio value to the changes in the force of mortality was equal to 0. For each product, the duration was estimated by comparing the base value of the product with the shocked value obtained by moving the vector of the force of mortality by a given low constant. It was the first research, which could be used by insurance companies to determine the optimal structure of the portfolio and analyse the differences between the current and optimal structure. This approach was further developed in next years, mainly by adding new elements to the existing model, such as stochastic mortality models, mortality forecasting models (e.g. LC model and CBD model mentioned earlier) or interest rate models.

The subject of the thesis and research hypotheses

The subject of the thesis is a presentation of an own model to determine the optimal structure of insurance portfolio to hedge against longevity risk. The model is conceptually close to the duration approach proposed by Wang et al. (2010), main differences are related to the assumptions adopted and computational methods used. Unlike in Wang et al. (2010), the approximate approach and computing effective duration was replaced by analytical approach (determination of the formula for optimal structure of the portfolio via analytical calculation

[6] S. H. Cox, Y. Lin, *Natural Hedging of Life and Annuity Mortality Risks*, North American Actuarial Journal 11, p. 1-15 (2007).

[7] J. L. Wang, H. C. Huang, S. S. Yang, J. T. Tsai, *An Optimal Product Mix for Hedging Longevity Risk in Life Insurance Companies: The Immunization Theory Approach*, The Journal of Risk and Insurance 77, p. 473-497 (2010).

of derivatives with regard to force of mortality), therefore the method proposed in the thesis was described as the analytical method. The actuarial present value of payments for each product, and consequently for the entire portfolio, was defined as a function of i.a. interest rate and force of mortality. In all calculations a flat and constant interest rate structure was used and the force of mortality was treated as a main variable impacting the value of the portfolio. Since the portfolio value was defined as a function with regard to multidimensional variable (force of mortality for each age group), additional assumption was made, that relative changes in the force of mortality were the same for each age group. It means that the force of mortality for a person at age x and in calendar year t was assumed at the level $\mu_x^t = k \cdot \mu_x^0$, where μ_x^0 stands for the force of mortality for a person at age x observed in the year when the determination of the optimal portfolio structure was performed. As a result of that, the portfolio value was a function of only one variable k. Next step was to find the portfolio structure, for which the derivative of this function with respect to k was equal to 0 (assuming the initial value of $k = 1$). The main differences in relation to the approach proposed by Wang et al. were therefore assuming multiplicative instead of additive changes in the force of mortality and performing analytical calculations instead of simulations to determine the sensitivities.

Research hypotheses

The following research hypotheses were verified in the thesis:

- There are possibilities for insurance companies to benefit from the natural immunization via following an appropriate structure of the insurance portfolio, so that potential losses on life annuities resulting from increasing life expectancy are offset by gains on life insurance policies.
- The use of the analytical method leads to better hedging against longevity risk than the duration approach due to more precise calculation of sensitivity with regard to force of mortality and adopting more realistic assumptions (multiplicative changes in the force of mortality seem more reasonable than additive changes when compared to the development of life tables for Poland published by GUS in the period 1990-2016).
- The use of the analytical method leads to a similar hedging effect when compared with the use of dynamic life tables estimated based on mortality forecasting models, such as Lee-Carter model or CBD model.

The research method and results

The above hypotheses were verified in the thesis with use of models simulating payments related to life annuities and life insurance policies in different scenarios. The calculations were performed mainly in MS Excel, additional programs, such as R or Gretl, were also used (e.g. to estimate parameters of LC and CBD models). The current section describes the main results of the analyses performed in the context of the hypotheses being verified.

Existence of the hedging effect against longevity risk resulting from the portfolio structure optimization

In order to prove the existence of the hedging effect against longevity risk a hypothetical example of insurance company selling only life insurance policies and life annuities for a single premium was analysed. The aim was to investigate how the actuarial present value of the payments resulting from these products changes in line with mortality tables development. The table presented below shows sample results for a situation where whole life insurance policies with sum assured equal to 100 000 PLN were sold to 40 years old women and whole life annuities with annual annuity payment equal to 10 000 PLN were sold to 60 years old women. The force of interest was equal to 5%, a single premium for each insurance product was calculated with use of the life tables published by GUS for the year 1990, the actual mortality however followed the life tables from a different year, e.g. 2000, 2010 and 2016.

Table 1: Sensitivity of the portfolio value to changes in life tables – sample analysis for females

	Life insurance	Life annuity	Portfolio
1990 life tables – premium received	1 403 141 498.17 zł	2 732 005 742.78 zł	4 135 147 240.95 zł
2000 life tables	1 290 893 400.99 zł	2 858 966 299.94 zł	4 149 859 700.93 zł
relative change	*-8.000%*	*4.647%*	*0.356%*
2010 life tables	1 162 206 986.81 zł	2 997 891 107.28 zł	4 160 098 094.08 zł
relative change	*-17.171%*	*9.732%*	*0.603%*
2016 life tables	1 094 084 709.33 zł	3 057 440 148.55 zł	4 151 524 857.88 zł
relative change	*-22.026%*	*11.912%*	*0.396%*

The above analysis shows that the change in life tables between year 1990 and year 2016 leads to the increase in actuarial present value of payments for life annuities by 11.9% compared to the premium received and the decrease in actuarial present value of payments for life insurance by 22.0% compared to the premium received. As expected, the impacts have opposite signs. The results for the balanced portfolio with the structure optimized using analytical approach show that the difference between the actuarial present value of payments for the entire portfolio compared to the premium received is only 0.4%. It is not a perfect hedge against changes in mortality, nevertheless this result indicates good level of immunization, which supports the hypothesis.

Similar analyses were performed for different combinations of the policyholders' entry age and also for males, leading to similar results.

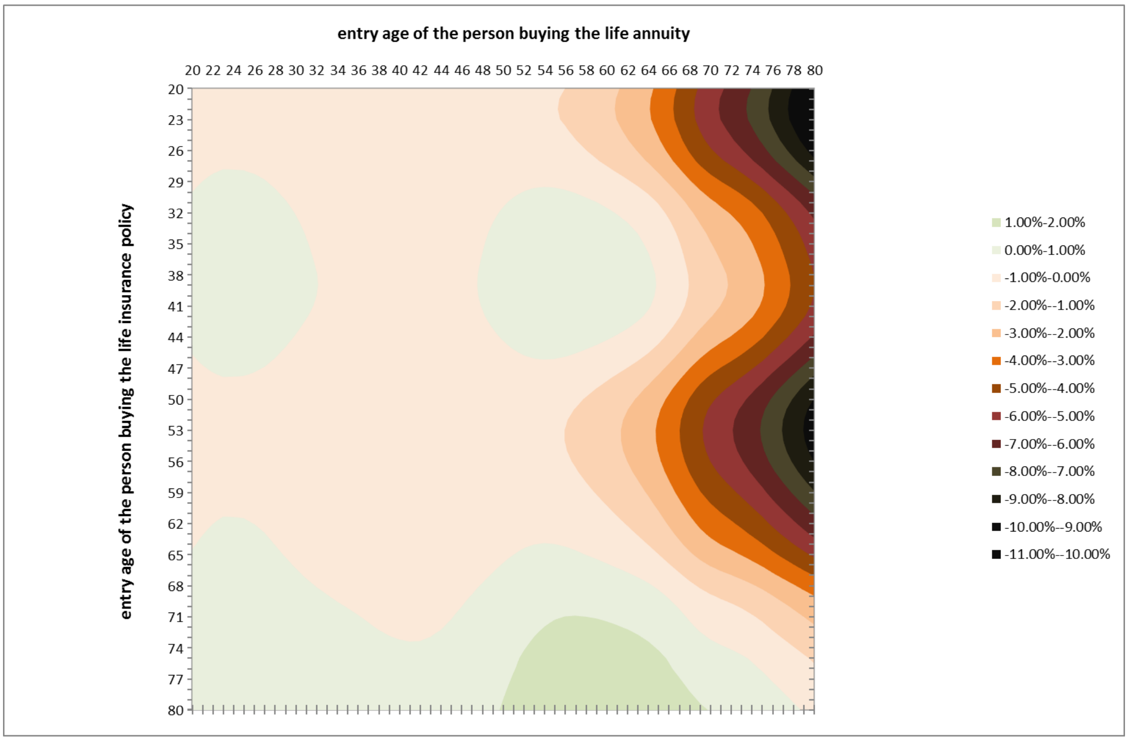
Comparison of the hedge efficiency between the analytical method and the duration approach

The comparison of the hedging efficiency between the analytical method and the duration approach was performed by means of simulating the impact of historical changes in mortality in Poland according to life tables published by GUS. In the constructed model the insurance company calculated the optimal structure of the portfolio based on Polish life tables for 1990 and sold life insurance policies and life annuities in line with this structure. Separately two cases were considered when the policyholders are males and females (due to different mortality coefficients). Also, different scenarios of entry age for the policyholders were analysed (between 20 and 80 years old). This gave jointly 3721 combination for males and 3721 combinations for females (61 possible values for the entry age of the buyer of life insurance policy and 61 possible values for the entry age of the buyer of life annuity). All policies were issued for the maximum period of 26 years or until the insured person turned 100 years old (the coverage period started on January 1, 1991 and ended on December 31, 2016 or if the insured person turned 100 years old). The aim of this approach was to use fully the available history of life tables published by GUS[8]. In the model, at the beginning of the coverage period the policyholder paid a single premium calculated with use of Polish life tables for the year 1990, in subsequent years the mortality among the policyholder followed the official Polish life tables published by GUS for a given year and was different than initially assumed, which led to lower or higher benefit payments compared to those expected at the moment of premium calculation. Due to a different character of the insurance products sold, the insurance company generated gains on life insurance policies (the decrease of mortality coefficients in subsequent years led to lower death payments), but suffered from losses on life annuities (increasing life expectancy for the annuitants led to increased period of

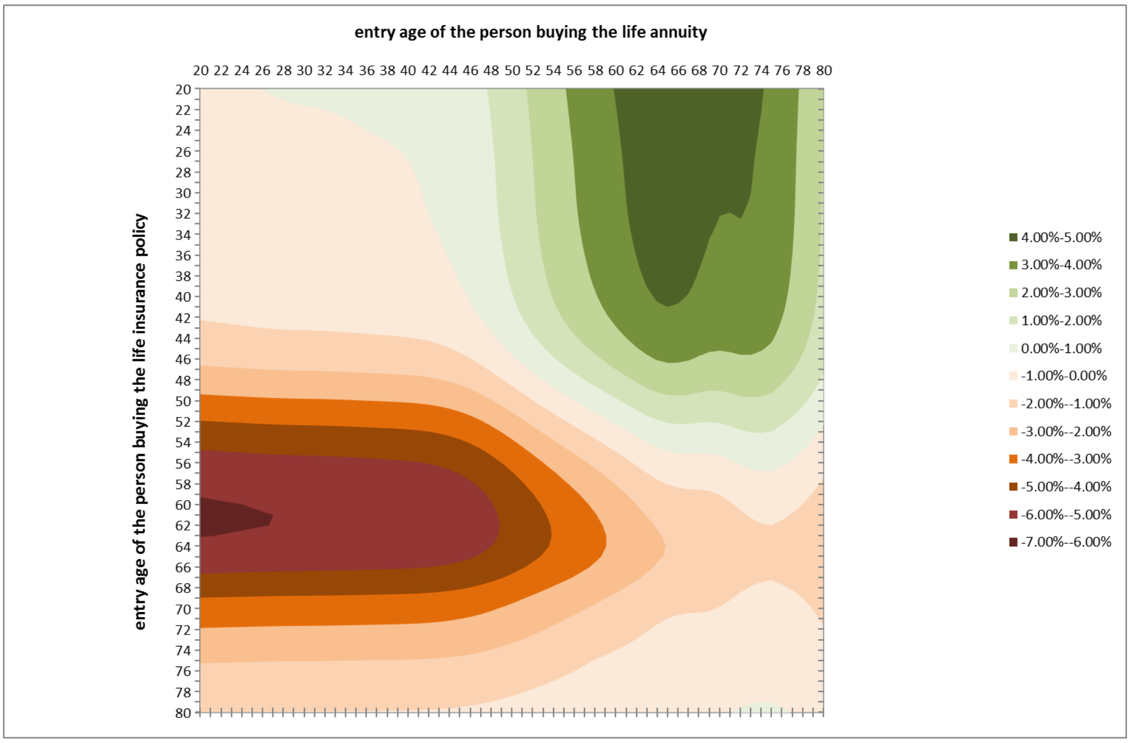
[8] There is longer time series of Polish life tables available on the website of Human Mortality Database (www.mortality.org), covering the period since 1958. In the thesis only the life tables published by GUS were used in order to maintain internal consistency.

annuity benefits payment). Two situations were considered in this model. In the first one, the structure of the portfolio (i.e. the number of life insurance and life annuity policies sold) was determined using the analytical method; in the second one, the duration approach was used. Thus, the model allowed for a direct comparison of both methods in relation to the historical changes of mortality in Poland and also allowed to compare financial consequences for the insurance company resulting from using each of these methods. For each age combination, separately in the case when the policies were sold to males and females, the deviation between the present value of the payments calculated with use of historical life tables and the actuarial present value of the payments expected at the moment of premium calculation was determined. The deviations for both methods are presented in the graphs below (the graphs show only the results for females, the analysis performed for males leads to similar results):

Graph 1. Sensitivity of the insurance portfolio value with regard to changes in mortality for the optimal portfolio structure, calculated with use of analytical method for females depending on age



Graph 2. Sensitivity of the insurance portfolio value with regard to changes in mortality for the optimal portfolio structure, calculated with use of duration approach for females depending on age



Graphs 1 and 2 illustrate the relative deviation between the present value of the payments settled by the insurance company and the premium received for different combinations of policyholders' entry age (the entry age of the person buying the life insurance policy is depicted on the vertical axis and the entry age of the person buying the life annuity is depicted on the horizontal axis). Different shades of red represent those combinations for which the present value of actual payments was lower than the premium received, while different shades of green represent those combinations for which the present value of actual payments was higher than the premium received. The darker the colour, the higher the relative deviation compared to the expected level. The brighter the colour, the lower the deviation and the better the portfolio hedge against the changes in the force of mortality. The comparison of the graphs above shows that the use of the analytical method leads more often to lower deviations than the use of duration approach (the share of the bright part on the graph is higher). It is also worth to analyse the quantitative results, such as maximum deviation *in plus*, maximum deviation *in minus* or average deviation (calculated as an average out of 3721 age combinations) presented in Table 2.

Table 2: Summary results for the portfolio sensitivity to the changes in life tables for females calculated for the analytical method and the duration approach

	Analytical method	Duration approach
Maximum deviation *in minus*	-10.35%	-6.23%
Maximum deviation *in plus*	1.79%	4.47%
Average deviation	2.35%	2.83%
Number of age combinations when the method is better	2 585	1 136

The maximum deviation *in minus* is higher in case of analytical method, however the average deviation is lower (2.35% compared to 2.83%), moreover for 2 585 age combinations (which constitutes 69.5% of all combinations analysed) the analytical method leads to lower absolute value of the deviation (thus to a better result) than the duration approach.

The better hedge efficiency when using the analytical method partially results from assuming multiplicative changes in the force of mortality, while under the duration approach additive changes are assumed. Table 3 shows that the relative changes in the force of mortality between year 1990 and 2016 are similar for different age groups, while the absolute changes clearly increase with age.

Table 3: Relative and absolute changes in mortality intensity between 1990 and 2016 depending on age for males and females

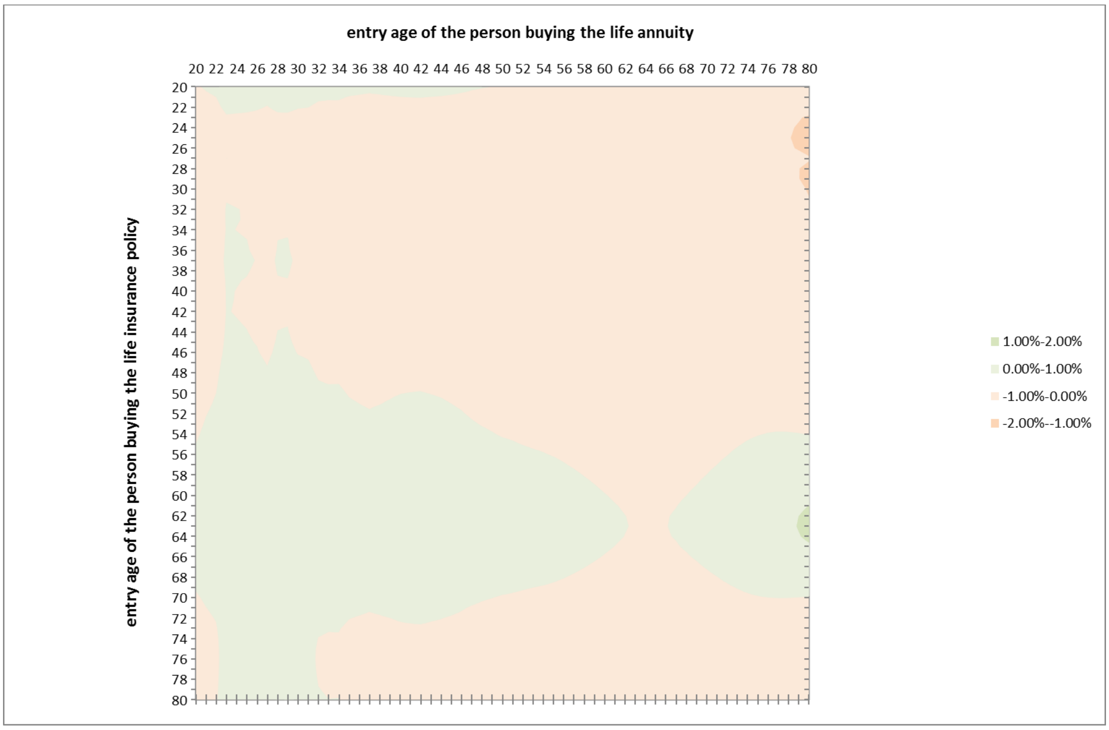
Age	Relative change in mortality intensity		Absolute change in mortality intensity	
	Males	Females	Males	Females
0	-79%	-79%	-0.0174	-0.0136
10	-63%	-73%	-0.0002	-0.0002
20	-48%	-42%	-0.0007	-0.0002
30	-48%	-48%	-0.0011	-0.0003
40	-50%	-53%	-0.0025	-0.0010
50	-43%	-39%	-0.0050	-0.0016
60	-36%	-35%	-0.0095	-0.0036
70	-37%	-45%	-0.0197	-0.0127
80	-40%	-47%	-0.0503	-0.0397
90	-37%	-38%	-0.1013	-0.0880
100	-37%	-36%	-0.2212	-0.2028

The above results confirm the hypothesis that the analytical method leads to a better hedging impact against longevity risk than the duration approach.

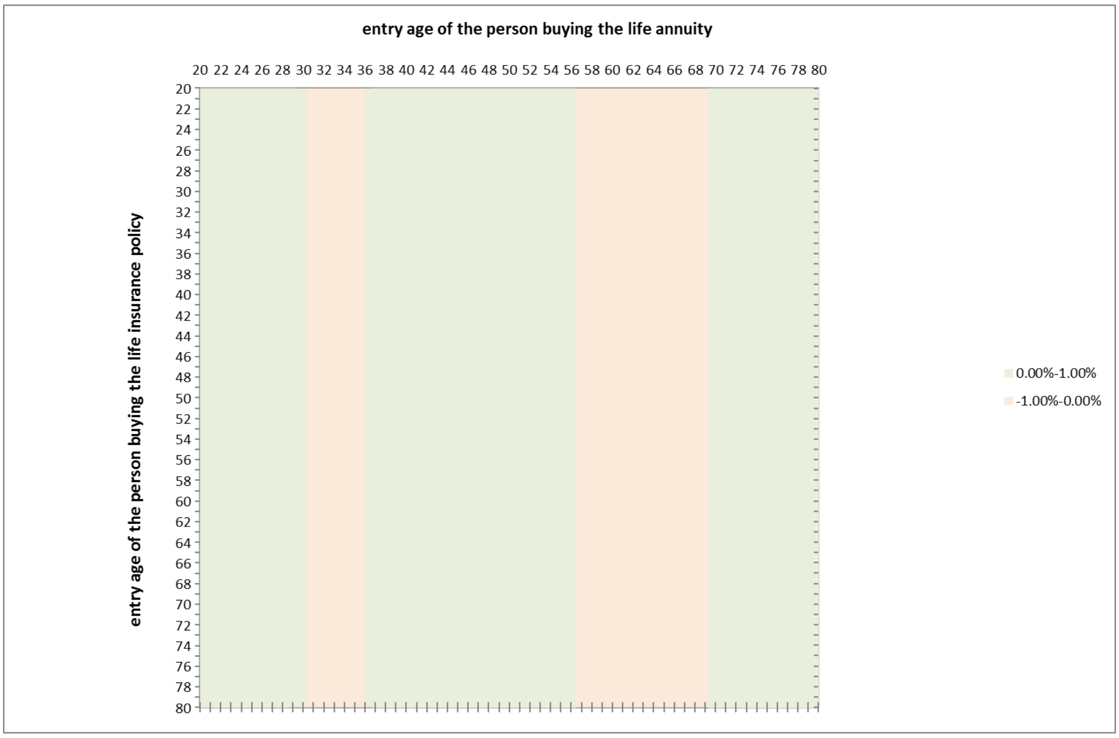
Comparison of the hedge efficiency between the analytical method and mortality forecasting models

The aim of the second simulation model was to compare the hedge efficiency resulting from the use of optimal structure of the insurance portfolio calculated using the analytical method and the hedge efficiency resulting from the application of mortality forecasting models (LC and CBD). In the first variant the insurance company determined the optimal structure of the portfolio with use of Polish life tables for the year 2009 and sold life insurance and life annuity policies in line with this structure. Similarly to the model previously described, 3721 age combinations, separately for males and for females were considered. The single premium was calculated based on the life tables for the year 2009. In the second and the third variant the company sold only life annuities and the insurance premium was calculated based on dynamic life tables estimated with use of LC model (the second variant) or CBD model (the third variant). The parameters of both models were estimated based on historical changes in life tables in years 1990-2009. All policies were sold for the period of 7 years (the coverage period lasted since January 1, 2010 till December 31, 2016). Thus, the model allowed for a direct comparison of the three hedging methods (optimal portfolio structure, LC model and CBD model) in relation to the historical changes of mortality in Poland and also allowed to compare financial consequences for the insurance company resulting from using each of these methods. For each age combination, separately in the case when the policies were sold to males and females, the deviation between the present value of the payments calculated with use of historical life tables and the actuarial present value of the payments expected at the moment of premium calculation was determined. The deviations for both methods are presented in Graphs 3, 4 and 5 (the graphs show only the results for females, the analysis performed for males led to similar results). Due to the fact that in the second and the third variant (using LC and CBD model) only life annuities were sold, the results did not depend on the entry age of the person buying the life insurance policy. They are still, however, presented on a two-dimensional graph in order to maintain consistency with the remaining graphs.

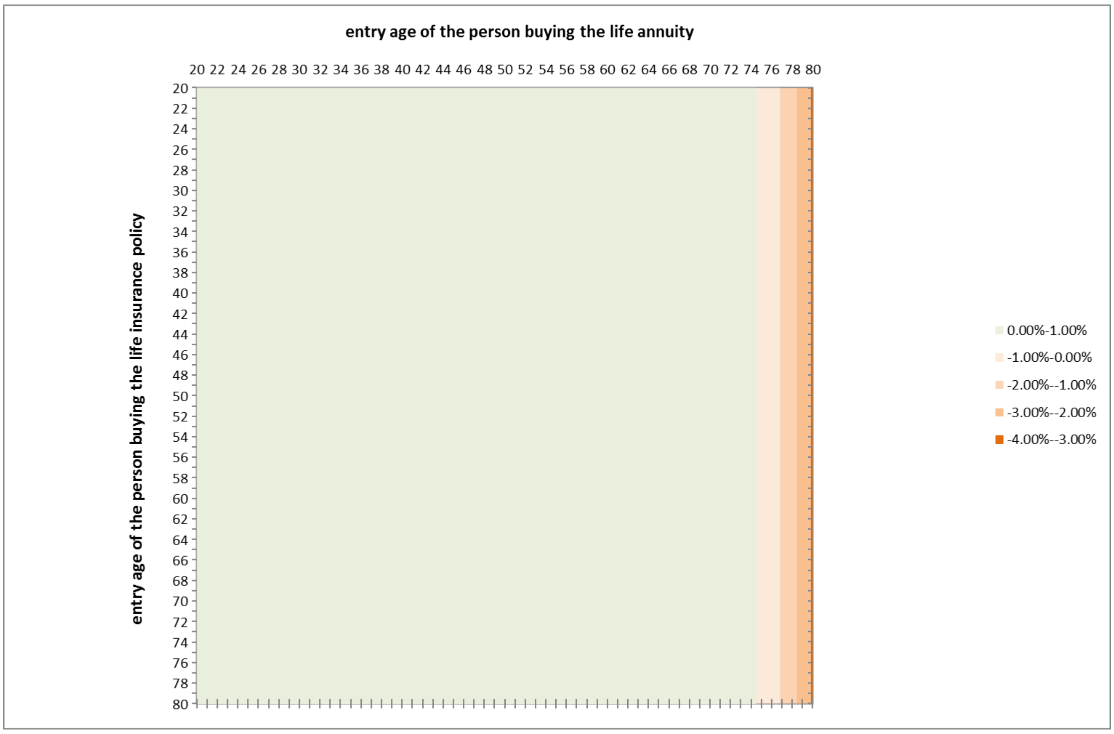
Graph 3. Sensitivity of the insurance portfolio value with regard to changes in mortality for the optimal portfolio structure, calculated with use of the analytical method for females depending on age



Graph 4. Sensitivity of the insurance portfolio value with regard to changes in mortality when using the dynamic life tables determined in line with the Lee-Carter model for females depending on age



Graph 5. Sensitivity of the insurance portfolio value with regard to changes in mortality when using the dynamic life tables determined in line with the CBD model for females depending on age



The analysis of the above graphs shows good hedge efficiency in case of the analytical method and the LC model (no dark area). The CBD model led generally to good results, except for the situation when the entry age of the woman buying the life annuity was close to 80. The quantitative results, similar to the previous model, are presented in the tables below. Table 4 shows the results for all 3 721 age combinations, while Table 5 is limited to the most representative age combinations only, i.e. when the entry age of the person buying life insurance policy is between 30 and 60 years, and the entry age of the person buying the life annuity is between 50 and 80 years (961 age combinations in total).

Table 4: The sensitivity of the insurance portfolio to the changes in life tables in the three analysed scenarios for females

	Optimal structure	**Lee-Carter model**	**CBD model**
Maximum deviation *in minus*	-1.21%	-0.25%	-3.14%
Maximum deviation *in plus*	1.10%	0.93%	0.65%
Average deviation	0.268%	0.273%	0.62%
Number of age combination when the method is the best	1 679	1 574	468

Table 5: The sensitivity of the insurance portfolio to the changes in life tables in the three analysed scenarios for females (limited to those age combinations when the entry age of the person buying the life insurance policy is between 30 and 60 years, and the entry age of the person buying the life annuity is between 50 and 80 years)

	Optimal structure	**Lee-Carter model**	**CBD model**
Maximum deviation *in minus*	-1.04%	-0.25%	-3.14%
Maximum deviation *in plus*	0.89%	0.93%	0.65%
Average deviation	0.377%	0.381%	0.86%
Number of age combination when the method is the best	326	396	239

The results show that the hedge efficiency achieved through the portfolio structure optimisation using the analytical method is similar to the hedge efficiency achieved through the application of the LC model. In both tables the average deviations are similar for both methods. The number of age combinations for which the given method was the best is highest for the analytical method when the entire sample of 3 721 age combinations is considered. When limited to 961 most representative age combinations, however, the LC method more often led to lower deviation. Compared with these two methods, the CBD model led to relatively worst results. Thus, the analysis performed confirmed the hypothesis that the hedge efficiency achieved through the portfolio optimization structure is comparable to the hedge efficiency achieved due to the use of mortality forecasting models.

Practical possibilities to use the results

The results obtained show that the analytical method is a good way to hedge against longevity risk and its quality is comparable to the commonly used forecasting models. The problem is that while a partial hedging due to the use of analytical method may occur autonomously in insurance portfolio comprising life annuities and life insurance policies, it is not easy to change the portfolio structure so that it is equal to the target structure leading to the full immunization. Therefore, the use of natural immunization by insurance companies is limited. Nevertheless, some areas where even a partial hedging can be beneficial for the companies can be identified.

First such an area is product pricing. Obviously, if the optimal structure of the portfolio estimated using the analytical method is for example 75% of life insurance policies and 25% of life annuities (looking at number of policies) it doesn’t mean that the insurance company can easily and in short term adapt its portfolio structure to the optimal level. It can however estimate its exposure to longevity risk, e.g. if there are more life annuities and less life insurance policies in the portfolio than would result from the optimal structure, it means that the decrease of mortality coefficients would lead to a loss, while in the opposite situation it would lead to a gain. Knowing that, the company can estimate what the cost of additional hedging against longevity risk (e.g. due to buying financial hedging instruments or simply holding additional risk capital) is and use this information in pricing. For example, if there are relatively too few life insurance policies in the portfolio, the company could decrease the premiums in order to enhance the demand for this product. Then, the loss resulting from the decrease in premiums could be offset by the gain resulting from the increased level of hedging against longevity risk.

The second area for a practical implementation of the natural immunization is the estimation of what part of the portfolio is unhedged and thus what the actual needs for additional hedging through for example financial instruments are. Usually, the share of life insurance policies in the portfolio of insurance companies selling mainly life annuity products is insufficient to hedge the entire portfolio against longevity risk. Thus, there is no possibility for the full hedging with use of natural immunization. Nevertheless, some hedging effect, at least partial, exists. In the case when the portfolio of the insurance company contains relatively too few life insurance policies, the company is able to estimate what portion of life annuity portfolio is hedged by them. It means that the company's actual exposure to longevity risk is lower than would normally result from the analysis of the entire portfolio. The company can use this information when purchasing additional hedging with use of financial instruments (such as longevity bonds or swaps). Since the additional hedging would apply only to a part of the entire portfolio (that part which is not hedged by life insurance policies), it would be cheaper than in the case when it is purchased for the entire portfolio.

Lastly, the analysis of the natural immunization of insurance portfolio can be used in risk management processes, since longevity risk is a crucial element of the risk management strategy. One obvious area to use it could be the calculation of Solvency II capital requirement with use of internal models. The information that due to natural immunization a part of the insurance company's portfolio is hedged against mortality changes can be used when determining the capital requirement for mortality and longevity risk. The partial hedge of the portfolio should decrease the exposure to longevity risk, thus leading to a decrease of solvency capital requirement and consequently generating a profit for the insurance company. Moreover, the knowledge about partial hedging can be used in other areas of risk management, e.g. when analysing risk appetite. The higher part of the portfolio is immunized, the more new policies can be issued within the risk appetite framework. This may lead to the increase of market share and also to the increase of the company's profit (assuming that the new policies are profitable).

Summary

The following can be concluded in relation to the research hypotheses stated at the beginning:

- It was proved that insurance companies can benefit from the use of natural immunization of insurance portfolio. Also, certain practical examples of the use of immunization by the companies were presented. Indeed, it is possible to optimize the portfolio structure in order to hedge the portfolio against longevity risk.
- The use of analytical method, proposed in the thesis, leads to a better hedge impact than the duration approach, which is a result of using more reasonable assumptions and which was proved by the analysis of the changes in Polish life tables published by GUS in the period 1990-2016.
- The hedge efficiency, achieved via portfolio structure optimization through the use of the analytical method, is comparable to the hedge efficiency obtained by the application of dynamic life tables estimated based on Lee-Carter model and better than in the case when CBD model is used.

To sum up, the thesis presents a new method of insurance portfolio optimization, which – as shown in the analysis of the historical data for Poland – leads to a good hedging against longevity risk and can be used, at least partially, in practice by insurance companies.

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